Efficient translation of sequent calculus proofs into natural deduction proofs

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Outline

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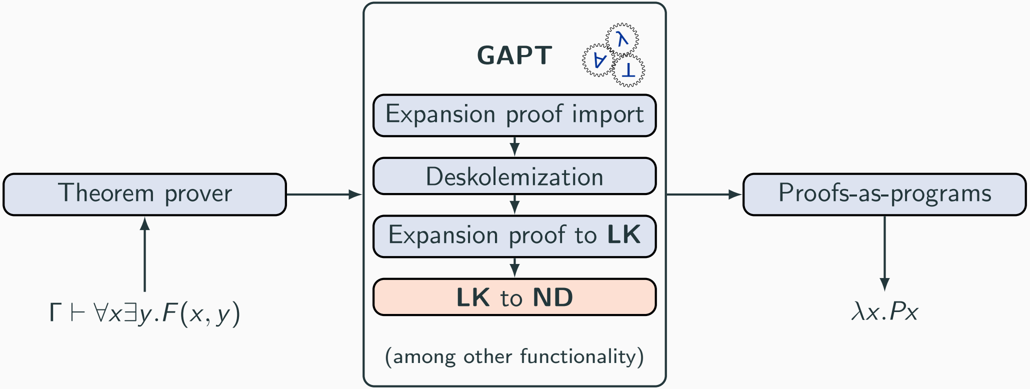
1. **Motivation**
2. Proof systems
3. Translation
4. Practical evaluation

Program extraction tool chain



Proof-as-programs: identity function

Theorem prover

$\vdash \forall x \exists y. x = y$

GAPT

$$\cfrac{\cfrac{\cfrac{}{\vdash v = v}\,(\text{refl})}{\vdash \exists y. v = y}\,(\exists{:}\text{r})}{\vdash \forall x. \exists y. x = y}\,(\forall{:}\text{r})$$

LK to **ND**

$$\cfrac{\cfrac{\cfrac{}{\vdash v = v}\,(=\,{:}\text{i})}{\vdash \exists y. v = y}\,(\exists{:}\text{i})}{\vdash \forall x. \exists y. x = y}\,(\forall{:}\text{i})$$

Proofs-as-programs

$$\cfrac{\cfrac{\cfrac{}{\vdash () : \text{Unit}}\,(=\,{:}\text{i})}{\vdash (v, ()) : (\text{Nat} \wedge \text{Unit})}\,(\exists{:}\text{i})}{\vdash \lambda v.(v, ()) : \text{Nat} \Rightarrow (\text{Nat} \wedge \text{Unit})}\,(\forall{:}\text{i})$$

Outline

Inference rules

Premisses

Antecedent Succedent

$$\frac{\Gamma \vdash \Delta, A \qquad \Sigma \vdash \Pi, B}{\Gamma, \Sigma \vdash \Delta, \Pi, A \wedge B}\ \text{Rule name}\ (\wedge{:}\mathrm{r})$$

Main formula

Conclusion

LK and ND

- Multi-succedent sequent calculus **LK** with cut
- Single-succedent natural deduction **ND** with a rule for excluded middle

LK

$$\frac{A, \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B} \, (\Rightarrow :\mathrm{r})$$

$$\frac{\Gamma \vdash \Delta, A \qquad B, \Sigma \vdash \Pi}{A \Rightarrow B, \Gamma, \Sigma \vdash \Delta, \Pi} \, (\Rightarrow :\mathrm{l})$$

ND

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \Rightarrow B} \, (\Rightarrow :\mathrm{i})$$

$$\frac{\Gamma \vdash A \Rightarrow B \qquad \Pi \vdash A}{\Gamma, \Pi \vdash B} \, (\Rightarrow :\mathrm{e})$$

Excluded middle rule of ND

$$\frac{\Gamma, A \vdash B \qquad \Pi, \neg A \vdash B}{\Gamma, \Pi \vdash B}\ (\mathrm{em})$$

Differences between LK and ND

LK

- Supports classical reasoning by allowing multiple formulas in the succedent
- Switches **focus** between formulas

$$\dfrac{\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg\text{:r})}{\vdash \boxed{A \lor \neg A}}\,(\lor\text{:r})$$

ND

- Only allows a single formula in the succedent

$$\dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\text{ax})}{A \vdash A \lor \neg A}\,(\lor\text{:i1}) \qquad \dfrac{\dfrac{}{\neg A \vdash \neg A}\,(\text{ax})}{\neg A \vdash A \lor \neg A}\,(\lor\text{:i2})}{\vdash A \lor \neg A}\,(\text{em})$$

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1. Motivation
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3. **Translation**
4. Practical evaluation

Prior work

- Previous translations either
  - used intermediate calculi [Gen35], or
  - worked for **LJ**, but did not give the translation for full **LK** [Zuc74, TS00], or
  - did not target **ND** [Gil17].

Main ideas

1. $\boxed{\textbf{LK} \text{ to } \textbf{ND}}$ translates each **LK** inference separately
2. **Focused** formula becomes single formula in succedent
3. Switching **focus** simulated using **exchange rule**

1. LK to ND translates each LK inference separately

$$\text{LK to ND}\left(\dfrac{\dfrac{\pi}{A[y/x], \Gamma \vdash B}}{\exists x A, \Gamma \vdash B}\,(\exists{:}\mathrm{l})\ , B \right) \stackrel{\text{def}}{=} \dfrac{\dfrac{}{\exists x A \vdash \exists x A}\,(\mathrm{ax}) \qquad \dfrac{\text{LK to ND}(\pi, B)}{A[y/x], \Gamma \vdash B}}{\exists x A, \Gamma \vdash B}\,(\exists{:}\mathrm{e})$$

1. $\boxed{\textbf{LK to ND}}$ translates each LK inference separately

$$\boxed{\textbf{LK to ND}}\left(\dfrac{\dfrac{\pi}{A[y/x], \Gamma \vdash B}}{\exists x A, \Gamma \vdash B}(\exists{:}\mathrm{l}), B \right) \stackrel{\text{def}}{=} \dfrac{\dfrac{}{\exists x A \vdash \exists x A}(\mathrm{ax}) \qquad \dfrac{\boxed{\textbf{LK to ND}}(\pi, B)}{A[y/x], \Gamma \vdash B}}{\exists x A, \Gamma \vdash B}(\exists{:}\mathrm{e})$$

- Derivations for each **LK** inference defined in the paper

2. Focused formula becomes single formula in succedent

LK to **ND**(π, A), where π is an **LK** proof, and $\boxed{A}$ is the **focused** formula.

$$\text{If} \quad \frac{\pi}{\Gamma \vdash \Delta, \boxed{A}} \quad \text{then} \quad \frac{\textbf{LK to ND}(\pi, A)}{\Gamma, \neg\Delta \vdash A}$$

2. Focused formula becomes single formula in succedent

LK to ND(π, A), where π is an **LK** proof, and $\boxed{A}$ is the **focused** formula.

$$\text{If} \quad \dfrac{\pi}{\Gamma \vdash \Delta, \boxed{A}} \quad \text{then} \quad \dfrac{\textbf{LK to ND}(\pi, A)}{\Gamma, \neg\Delta \vdash A}$$

$$\text{and if} \quad \dfrac{\pi}{\Gamma \vdash} \text{, then} \quad \dfrac{\textbf{LK to ND}(\pi, -)}{\Gamma \vdash \bot} .$$

3. Switching focus simulated using exchange rule

Exchange macro rule

$$\frac{\Gamma, \neg A \vdash B}{\Gamma, \neg B \vdash A}\ (\text{ex}) \quad \stackrel{\text{def}}{=} \quad \dfrac{\dfrac{}{A \vdash A}\ (\text{ax}) \qquad \dfrac{\dfrac{\dfrac{}{\neg B \vdash \neg B}\ (\text{ax}) \qquad \Gamma, \neg A \vdash B}{\Gamma, \neg B, \neg A \vdash \bot}\ (\neg\text{:e})}{\Gamma, \neg B, \neg A \vdash A}\ (\bot\text{:e})}{\Gamma, \neg B \vdash A}\ (\text{em})$$

3. Switching focus simulated using exchange rule

$$\textbf{LK to ND}\left(\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r})}{\neg\neg A \vdash \boxed{A}}\,(\neg{:}\text{l})}{\vdash \boxed{\neg\neg A \Rightarrow A}}\,(\Rightarrow{:}\text{r}) \quad , \neg\neg A \Rightarrow A \right) \quad = \quad \dfrac{\dfrac{\textbf{LK to ND}\left(\dfrac{\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r})}{\neg\neg A \vdash \boxed{A}}\,(\neg{:}\text{l}) \quad , A \right)}{\neg\neg A \vdash A}}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})$$

3. Switching focus simulated using exchange rule

$$\textbf{LK to ND}\left(\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\mathrm{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\mathrm{r})}{\neg\neg A \vdash \boxed{A}}\,(\neg{:}\mathrm{l})}{\vdash \boxed{\neg\neg A \Rightarrow A}}\,(\Rightarrow{:}\mathrm{r}),\ \neg\neg A \Rightarrow A \right) = \begin{array}{c} \textbf{LK to ND}\left(\dfrac{\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\mathrm{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\mathrm{r})}{\neg\neg A \vdash \boxed{A}}\,(\neg{:}\mathrm{l}),\ A \right) \\ \hdashline \dfrac{\neg\neg A \vdash A}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\mathrm{i}) \end{array}$$

$$= \begin{array}{c} \textbf{LK to ND}\left(\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\mathrm{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\mathrm{r}),\ A \right) \\ \hdashline \dfrac{\neg\neg A \vdash A}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\mathrm{i}) \end{array} =$$

3. Switching focus simulated using exchange rule

$$\textbf{LK to ND}\left(\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r})}{\neg\neg A \vdash \boxed{A}}\,(\neg{:}\text{l})}{\vdash \boxed{\neg\neg A \Rightarrow A}}\,(\Rightarrow{:}\text{r}) \;, \neg\neg A \Rightarrow A \right) = \cfrac{\textbf{LK to ND}\left(\cfrac{\cfrac{\cfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r})}{\neg\neg A \vdash \boxed{A}}\,(\neg{:}\text{l}) \;, A \right)}{\cfrac{\neg\neg A \vdash A}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})}$$

$$= \cfrac{\textbf{LK to ND}\left(\cfrac{\cfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r}) \;, A \right)}{\cfrac{\neg\neg A \vdash A}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})} = \cfrac{\textbf{LK to ND}\left(\cfrac{\cfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r}) \;, \neg A \right)}{\cfrac{\cfrac{\neg A \vdash \neg A}{\neg\neg A \vdash A}\,(\text{ex})}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})}$$

Optimizations

Optimization 1: Stopping early

$$\boxed{\textbf{LK} \text{ to } \textbf{ND}}\left(\dfrac{\vdots\ \pi\ \vdots}{\vdash A, \neg A} \ , \neg A \right) \stackrel{\text{def}}{=} \dfrac{}{\neg A \vdash \neg A}\ (\text{ax})$$

Optimizations

Optimization 1: Stopping early

$$\boxed{\textbf{LK to ND}}\left(\dfrac{\pi}{\vdash A, \neg A}\;, \neg A\right) \overset{\text{def}}{=} \dfrac{}{\neg A \vdash \neg A}\,(\text{ax})$$

Translation of DNE

$$\dfrac{\dfrac{\boxed{\textbf{LK to ND}}\left(\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r})\;, \neg A\right)}{\dfrac{\neg A \vdash \neg A}{\neg\neg A \vdash A}\,(\text{ex})}}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})$$

$$=$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\boxed{\textbf{LK to ND}}\left(\dfrac{}{A \vdash A}\,(\text{ax})\;, A\right)}{A \vdash A} \quad \dfrac{}{\neg A \vdash \neg A}\,(\text{ax})}{\neg A, A \vdash \bot}\,(\neg{:}\text{e})}{\neg A \vdash \neg A}\,(\neg{:}\text{i})}{\neg\neg A \vdash A}\,(\text{ex})}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})$$

Optimizations

Optimization 1: Stopping early

$$\boxed{\textbf{LK to ND}}\left(\genfrac{}{}{0pt}{}{\pi}{\vdash A, \neg A}, \neg A \right) \stackrel{\text{def}}{=} \dfrac{}{\neg A \vdash \neg A}\,(\text{ax})$$

Translation of DNE

$$\dfrac{\boxed{\textbf{LK to ND}}\left(\dfrac{\dfrac{}{A \vdash \boxed{A}}\,(\text{ax})}{\vdash A, \boxed{\neg A}}\,(\neg{:}\text{r}), \neg A \right)}{\dfrac{\dfrac{\neg A \vdash \neg A}{\neg\neg A \vdash A}\,(\text{ex})}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})} = \dfrac{\dfrac{\dfrac{}{\neg A \vdash \neg A}\,(\text{ax})}{\neg\neg A \vdash A}\,(\text{ex})}{\vdash \neg\neg A \Rightarrow A}\,(\Rightarrow{:}\text{i})$$

Optimizations

Translation of $(\vee{:}\mathrm{r})$

$$\boxed{\textbf{LK}\text{ to }\textbf{ND}}\left(\dfrac{\dfrac{\pi}{\Gamma \vdash \Delta, A, B}}{\Gamma \vdash \Delta, A \vee B}\,(\vee{:}\mathrm{r})\ , A \vee B \right) \stackrel{\text{def}}{=} \dfrac{\dfrac{\dfrac{}{A \vdash A}\,(\mathrm{ax})}{A \vdash A \vee B}\,(\vee{:}\mathrm{i1}) \qquad \dfrac{\dfrac{\boxed{\textbf{LK}\text{ to }\textbf{ND}}(\pi, B)}{\neg A, \Gamma, \neg\Delta \vdash B}}{\neg A, \Gamma, \neg\Delta \vdash A \vee B}\,(\vee{:}\mathrm{i2})}{\Gamma, \neg\Delta \vdash A \vee B}\,(\mathrm{em})$$

Optimization 2: Constructive translation of $(\vee{:}\mathrm{r})$

$$\boxed{\textbf{LK}\text{ to }\textbf{ND}}\left(\dfrac{\dfrac{\dfrac{\pi}{\Gamma \vdash \Delta, A}}{\Gamma \vdash \Delta, A, B}\,(\mathrm{w{:}r})}{\Gamma \vdash \Delta, A \vee B}\,(\vee{:}\mathrm{r})\ , A \vee B \right) \stackrel{\text{def}}{=} \dfrac{\dfrac{\boxed{\textbf{LK}\text{ to }\textbf{ND}}(\pi, A)}{\Gamma, \neg\Delta \vdash A}}{\Gamma, \neg\Delta \vdash A \vee B}\,(\vee{:}\mathrm{i1})$$

Applies to 43.5% of $(\vee{:}\mathrm{r})$ inferences in our experiments.

Properties of the translation

Lemma

LK to **ND** *is linear in the size of the* **LK** *proof.*

Lemma

LK to **ND** *translates constructive* **LK** *proofs*[1] *into constructive* **ND** *proofs.*

[1] Basically Gentzen's **LJ** with different ($\vee$:r) rule

Outline

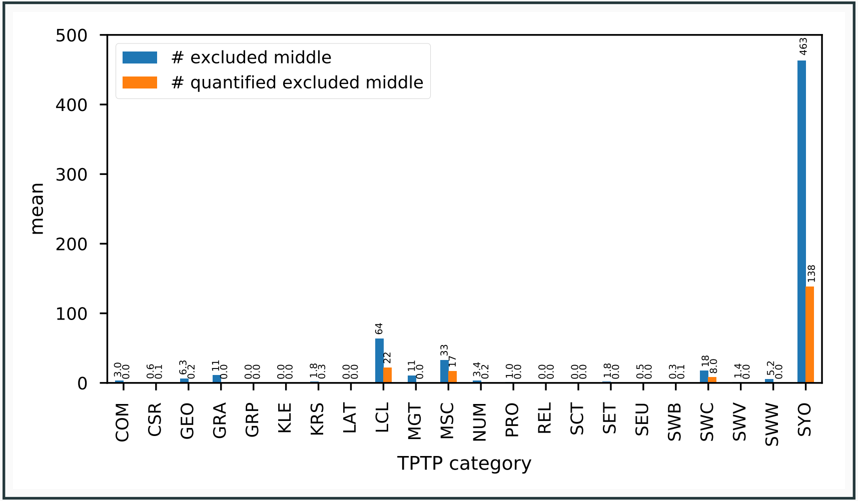
Practical evaluation

- 662 problems from CASC-26 (file size < 1 MB)
- 647 after clausification
- 291 proofs returned by E (2 minute timeout)
- 283 expansion proofs
- **261 LK proofs**
- **261 ND proofs** (on average $2.9\times$ larger)

Relative runtime of phases during proof import to natural deduction

Parser	CNF	E	replay	$\to$ exp.	deskolem.	$\to$ **LK**	$\to$ **ND**
2.95%	3.58%	50.49%	12.50%	4.28%	7.89%	17.66%	**0.65%**

Practical evaluation



Summary

- Toolchain for program extraction from automated proofs
- We presented **LK** to **ND** filling the last gap in the chain
  - Preserves local shape of the proof
  - Focusing approach using exchange
  - Simple Optimizations
  - Linear in the size of the **LK** proof
  - Maintains constructiveness
- Performs well in practice, introduces few excluded middle inferences

Implementation available at `https://github.com/gapt/gapt`

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