Complexity of decision problems on TRATGs

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Totally rigid acyclic tree grammars

Complexity

Cut-reduction

TRATGs

Terms, not words.

- Start symbol: A
- Nonterminals: $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \ldots$
- (Acyclic) productions: $\mathrm{B} \to t[\mathrm{C}, \mathrm{D}, \ldots]$

Rigid derivations: $\mathrm{A}[\mathrm{A}\backslash t_1][\mathrm{B}\backslash t_2][\mathrm{C}\backslash t_3] \cdots$

Language $L(G)$ consists of all derivable terms

TRATG example

$$\mathrm{A} \rightarrow \mathsf{f}(\mathrm{B}, \mathrm{B}) \mid \mathsf{g}(\mathrm{B}, \mathrm{B})$$
$$\mathrm{B} \rightarrow \mathsf{c} \mid \mathsf{d}$$

TRATG example

$$\mathrm{A} \to \mathsf{f}(\mathrm{B}, \mathrm{B}) \mid \mathsf{g}(\mathrm{B}, \mathrm{B})$$
$$\mathrm{B} \to \mathsf{c} \mid \mathsf{d}$$

$$L(G) = \{\mathsf{f}(\mathsf{c}, \mathsf{c}), \mathsf{f}(\mathsf{d}, \mathsf{d}), \mathsf{g}(\mathsf{c}, \mathsf{c}), \mathsf{g}(\mathsf{d}, \mathsf{d})\}$$

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Membership

Problem (Membership)
Given a TRATG G and a term t, is $t \in L(G)$?

Membership

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Claim: Membership is NP-complete.

- Derivations of t are polynomial in the size of t and G.

$$\mathrm{A},\ \mathrm{A}[\mathrm{A}\backslash s_1],\ \mathrm{A}[\mathrm{A}\backslash s_1][\mathrm{B}\backslash s_2],\ \ldots$$

  Can check in polynomial time whether such a sequence of terms is a derivation of t in G.

- Hardness: next slide.

Encoding SAT

The TRATG $\mathrm{Sat}_{n,m}$ generates the satisfiable 3-CNFs with n clauses and m variables:

$$\begin{aligned}
\mathrm{A} &\to \mathtt{and}(\mathrm{Clause}_1, \ldots, \mathrm{Clause}_n) \\
\mathrm{Clause}_i &\to \mathtt{or}(\mathrm{True}_i, \mathrm{Any}_{i,1}, \mathrm{Any}_{i,2}) \\
\mathrm{Clause}_i &\to \mathtt{or}(\mathrm{Any}_{i,1}, \mathrm{True}_i, \mathrm{Any}_{i,2}) \\
\mathrm{Clause}_i &\to \mathtt{or}(\mathrm{Any}_{i,1}, \mathrm{Any}_{i,2}, \mathrm{True}_i) \\
\mathrm{Any}_{i,k} &\to \mathtt{x}_1 \mid \mathtt{neg}(\mathtt{x}_1) \mid \cdots \mid \mathtt{x}_m \mid \mathtt{neg}(\mathtt{x}_m) \mid \mathtt{false} \mid \mathtt{true} \\
\mathrm{True}_i &\to \mathrm{Value}_1 \mid \cdots \mid \mathrm{Value}_m \mid \mathtt{true} \\
\mathrm{Value}_j &\to \mathtt{x}_j \mid \mathtt{neg}(\mathtt{x}_j)
\end{aligned}$$

Containment

Problem (Containment)
Given TRATGs G_1 and G_2, is $L(G_1) \subseteq L(G_2)$?

Containment

Problem (Containment)
Given TRATGs G_1 and G_2, is $L(G_1) \subseteq L(G_2)$?

Claim: Π_2^P-complete

- In Π_2^P: for every sequence of terms check if it is a derivation of a term t in G_1, and then if $t \in L(G_2)$.

Containment (Π_2^P-hardness)

- Determining the truth of the quantified Boolean formula $\forall y_1 \ldots \forall y_k \exists x_1 \ldots \exists x_m\, f$ is Π_2^P-complete.
- Let f be in 3-CNF with n clauses.

- Is $f\sigma$ satisfiable for any $\sigma \colon \{y_1, \ldots, y_k\} \to \{\mathtt{true}, \mathtt{false}\}$?
- Is $\{f\sigma \mid \sigma \colon \{y_1, \ldots, y_k\} \to \{\mathtt{true}, \mathtt{false}\}\} \subseteq L(\mathrm{Sat}_{n,m})$?
- Left side is generated by a TRATG:

$$\mathrm{A} \to f[y_1 \backslash \mathrm{Y}_1, \ldots, y_k \backslash \mathrm{Y}_k]$$
$$\mathrm{Y}_j \to \mathtt{true} \mid \mathtt{false}$$

Other problems

Problem (Disjointness)
Given TRATGs G_1 and G_2, is $L(G_1) \cap L(G_2) = \emptyset$?

Problem (Equivalence)
Given TRATGs G_1 and G_2, is $L(G_1) = L(G_2)$?

$\Rightarrow$ Disjointness is coNP-complete (via Membership)

$\Rightarrow$ Equivalence is Π_2^P-complete (via Containment)

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Proofs with Π_1-cuts

Definition (simple proof)
We call a proof π in LK simple iff:

- The end-sequent is prenex Σ_1
- Cuts have at most a single quantifier, which is prenex
- Quantified cuts are immediately followed by a strong quantifier rule

Cut-reduction to language generation

We assign to every simple proof π a TRATG $G(\pi)$.

$L(G(\pi))$ contains the formulas in a Herbrand sequent of π

- Nonterminals: eigenvariables from cuts + start symbol A
- Productions $x \to t$ for weak quantifier inferences on cut formulas:

$$\dfrac{\dfrac{\vdash \varphi(x)}{\vdash \forall x\, \varphi(x)}\,\forall\text{-r} \qquad \dfrac{\dfrac{\varphi(t) \vdash}{\vdots}\,\forall\text{-l}}{\forall x\, \varphi(x) \vdash}}{}\,\text{cut}$$

- Productions $\mathrm{A} \to \varphi(t)$ for instances of formulas end-sequent.

Herbrand-confluence

Theorem ([Hetzl and Straßburger 2012])

- *For every Gentzen cut-reduction sequence* $\pi \rightsquigarrow \pi'$*, we have* $L(G(\pi)) \supseteq L(G(\pi'))$.
- *If we did not perform grade reduction on weakenings, then* $L(G(\pi)) = L(G(\pi'))$.

Let $\overset{ne}{\rightsquigarrow}$ be the *non-erasing* Gentzen cut-reduction relation, i.e. where we do not reduce weakenings.

We can directly extract tautological Herbrand sequents from $\overset{ne}{\rightsquigarrow}$-NFs.

$\Rightarrow f \in H(\pi^*)$ iff $f \in L(G(\pi))$ (for any $\overset{ne}{\rightsquigarrow}$-NF π^*)

Corresponding problems for simple proofs

Problem (H-membership)
Let π *be a simple proof, and* f *a formula. Is there a* $\overset{ne}{\rightsquigarrow}$*-NF* $\pi \overset{ne}{\rightsquigarrow} \pi^*$ *such that* $f \in H(\pi^*)$*?*

Problem (H-containment)
Let π_1, π_2 *be simple proofs. Are there* $\overset{ne}{\rightsquigarrow}$*-NFs* $\pi_i \overset{ne}{\rightsquigarrow} \pi_i^*$ *such that that* $H(\pi_1^*) \subseteq H(\pi_2^*)$*?*

Problem (H-disjointness)
Let π_1, π_2 *be simple proofs. Are there* $\overset{ne}{\rightsquigarrow}$*-NFs* $\pi_i \overset{ne}{\rightsquigarrow} \pi_i^*$ *such that* $H(\pi_1^*) \cap H(\pi_2^*) = \emptyset$*?*

Problem (H-equivalence)
Let π_1, π_2 *be simple proofs. Are there* $\overset{ne}{\rightsquigarrow}$*-NFs* $\pi_i \overset{ne}{\rightsquigarrow} \pi_i^*$*, such that that* $H(\pi_1^*) = H(\pi_2^*)$*?*

Language generation to cut-reduction

Lemma

There is a formula $\varphi(x)$ such that we can assign to every grammar G a simple proof π_G satisfying $H(\pi_G^*) = \varphi[L(G)]$ for any $\overset{ne}{\rightsquigarrow}$-NF π_G^*.

Language generation to cut-reduction

Lemma
There is a formula $\varphi(x)$ such that we can assign to every grammar G a simple proof π_G satisfying $H(\pi_G^*) = \varphi[L(G)]$ for any $\overset{ne}{\rightsquigarrow}$-NF π_G^*.

Set $\varphi(x) := L(x) \to L(x)$.

Let $x_0, x_1, \ldots, x_n$ be the nonterminals of G, and $x_i \to t_{i,1} \mid \cdots \mid t_{i,k_i}$ the productions.

$$
\dfrac{
\dfrac{\dfrac{}{\vdash \varphi(t_{n,1}), \ldots, \varphi(t_{n,k_0})}}{\vdash \exists x\, \varphi(x)}
\qquad
\dfrac{\dfrac{\dfrac{\vdots\quad \dfrac{\dfrac{}{\vdash \varphi(t_{0,1}), \ldots, \varphi(t_{0,k_n})}}{\vdash \exists x\, \varphi(x)}}{}}{\varphi(x_n) \vdash \exists x\, \varphi(x)}}{\exists x\, \varphi(x) \vdash \exists x\, \varphi(x)}
}{\vdash \exists x\, \varphi(x)}\ \text{cut}
$$

Corresponding complexity results for simple proofs

Problem (H-membership)
Let π be a simple proof, and f a formula. Is there a $\overset{ne}{\rightsquigarrow}$-NF $\pi \overset{ne}{\rightsquigarrow} \pi^*$ such that $f \in H(\pi^*)$?

$\Rightarrow$ NP-complete

Problem (H-containment)
Let π_1, π_2 be simple proofs. Are there $\overset{ne}{\rightsquigarrow}$-NFs $\pi_i \overset{ne}{\rightsquigarrow} \pi_i^*$ such that that $H(\pi_1^*) \subseteq H(\pi_2^*)$?

$\Rightarrow$ Π_2^P-complete

Corresponding complexity results for simple proofs

Problem (H-disjointness)
Let π_1, π_2 be simple proofs. Are there $\overset{ne}{\rightsquigarrow}$-NFs $\pi_i \overset{ne}{\rightsquigarrow} \pi_i^*$ such that $H(\pi_1^*) \cap H(\pi_2^*) = \emptyset$?

$\Rightarrow$ coNP-complete

Problem (H-equivalence)
Let π_1, π_2 be simple proofs. Are there $\overset{ne}{\rightsquigarrow}$-NFs $\pi_i \overset{ne}{\rightsquigarrow} \pi_i^*$, such that that $H(\pi_1^*) = H(\pi_2^*)$?

$\Rightarrow$ Π_2^P-complete

Conclusion

- We can analyze cut-reduction using tree grammars.

Future work:

- Given a set of terms T and $n \geq 0$, is there a TRATG G such that $T \subseteq L(G)$ with at most n productions?
    - NP-complete if G has two nonterminals, otherwise unknown.